MAT251 Multivariable Calculus III Test #3 Name Robert Albertson
Prof. PORTER Spring 2012

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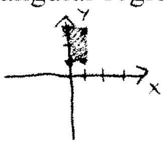
1. Set up a the triple integral to find the volume under the function **f(x,y) = xy + 2x +100** and above the rectangular region between the points (0,1),(1,1)(0,3),(1,3).

Region:



$$V = \int_0^1 \int_1^3 \int_0^{xy+2x+100} 1 \, dz\, dy\, dx$$

Bounds

$0 \leq x \leq 1$

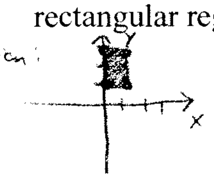
$1 \leq y \leq 3$

$0 \leq z \leq xy + 2x + 100$

Answer $V = \int_0^1 \int_1^3 \int_0^{xy+2x+100} dz\, dy\, dx$

2. Set up a the integral to find the surface area of the function **f(x,y) = xy + 2x +100** above the rectangular region between the points (0,1),(1,1)(0,3),(1,3)

Region:



$$S_A = \iint_D \sqrt{1 + \left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2}\, dA$$

$z = xy + 2x + 100$

$$S_A = \int_0^1 \int_1^3 \sqrt{1 + (y+2)^2 + x^2}\, dy\, dx$$

$\frac{\delta z}{\delta x} = y + 2$

$\frac{\delta z}{\delta y} = x$

Answer ↓

3. Set up a the triple integral to find the mass for the region under the function **f(x,y) = xy + 2x +100** and above the rectangular region between the points (0,1),(1,1)(0,3),(1,3) when the density is xy + 4

$P(x,y,z) = xy + 4$ ← density function

$$m = \iiint_E P(x,y,z)\, dV = \int_0^1 \int_1^3 \int_0^{xy+2x+100} xy + 4 \, dz\, dy\, dx$$

Answer $m = \int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy + 4 \, dz\, dy\, dx$

$$m = \int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy+4 \, dz\,dy\,dx$$ mass value | Put vertical on back

4. Give the integrals to find the center of gravity for the region under the function **f(x,y) = xy + 2x +100** and above the rectangular region between the points (0,1),(1,1)(0,3),(1,3) when the density is xy + 4

x multiplied the integrand

① $$\bar{x} = \frac{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} x^2y + 4x \, dz\,dy\,dx}{m}$$

* The "m" value in the denominator for $\bar{x}, \bar{y}, \bar{z}$ is the same value for m found in question 3.

② $$\bar{y} = \frac{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy^2 + 4y \, dz\,dy\,dx}{m}$$

y multiplied the integrand

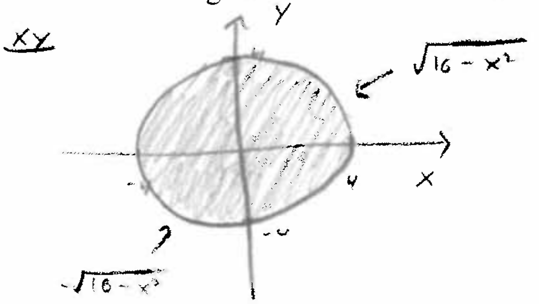
③ $$\bar{z} = \frac{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} zxy + 4z \, dz\,dy\,dx}{m}$$

z multiplied the integrand

Answer $(\bar{x}, \bar{y}, \bar{z})$; $\bar{x}, \bar{y}, \bar{z}$ are equal to the value given in ①, ②, ③ respectively.

coordinates of center of gravity

5. Set up the integral to find the volume for the region under the function **f(x,y) = xy + 2x +100** and above the circular region of radius 4 and centered at the origin using rectangular coordinates.



$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} xy + 2x + 100 \, dy\,dx$$

or

Answer $$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{xy+2x+100} 1 \, dz\,dy\,dx$$

though, they are equal.

6. Find the volume for the region under the function **f(x,y) = xy + 2x +100** and above the circular region of radius 4 and centered at the origin using polar coordinates.

let $x = r\cos\theta$

$y = r\sin\theta$

So $$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} f(x,y)\,dy\,dx = \int_0^{2\pi} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Answer 1600π work on back

$$= \int_0^{2\pi} \int_0^4 \left[(r\cos\theta)(r\sin\theta) + 2(r\cos\theta) + 100\right] r \, dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^4 r^3\cos\theta\sin\theta + 2r^2\cos\theta + 100r \, dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} r^4 \cos\theta\sin\theta + \frac{2}{3} r^3 \cos\theta + 50r^2 \Big|_0^4 \, d\theta =$$

continued on back

Problem 6

Continued

last step rewritten:

$$\int_0^{2\pi} \frac{1}{4} r^4 \cos\theta \sin\theta + \frac{2}{3} r^3 \cos\theta + 50r^2 \Big|_0^4 \, d\theta$$

$$= \int_0^{2\pi} 64 \cos\theta \sin\theta + \frac{128}{3} \cos\theta + 800 \, d\theta$$

$$= \int_0^{2\pi} 64\left(\tfrac{1}{2} \sin 2\theta\right) d\theta + \int_0^{2\pi} \frac{128}{3} \cos\theta + 800 \, d\theta$$

$$= -32\left(\frac{1}{2}\right)\cos(2\theta)\Big|_0^{2\pi} + \frac{128}{3} \sin\theta + 800\theta \Big|_0^{2\pi}$$

$$= -16\left[\cos 4\pi - \cos 0\right] + \frac{128}{3} \sin 2\pi + 1600\pi - \left(\frac{128}{3} \sin 0 + 0\right)$$

$$= -16(0) + 0 + 1000\pi - (0 + 0) = 1600\pi$$

Problem 4

$$\bar{x} = \frac{\int_0^1 \int_1^2 \int_0^{(xy+2x+100)} x^2y + 4x \, dz\,dy\,dx}{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy + 4 \, dz\,dy\,dx}$$

$$\bar{z} = \frac{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} zxy + 4z \, dz\,dy\,dx}{\int_0^1 \int_1^2 \int_0^{(xy+2x+100)} xy + 4 \, dz\,dy\,dx}$$

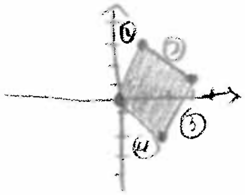
$$\bar{y} = \frac{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy^2 + 4y \, dz\,dy\,dx}{\int_0^1 \int_1^3 \int_0^{(xy+2x+100)} xy + 4 \, dz\,dy\,dx}$$

7. Evaluate the integral below by first using a change of variables, finding the jacobian, and then solving.

$\iint_R (3x-2y)dA$ when x=u+2v and y=3u-2v and the region is between

the points (0,0),(3,1),(2,-2),(1,3)

in xy plane we have:



found using the given points.

Associated equations with xy region are

① $y = 3x$

② $y = -x + 4$

③ $y = 3x - 8$

④ $y = -x$

Found using point-slope formula

See back of paper for work

we have

$$\int_0^1\int_0^1 -24u + 80v\,du\,dv = \int_0^1 -12 + 80v\,du$$

$$= -12 + 40 = 2$$

Answer 28

8. Change to spherical coordinates.

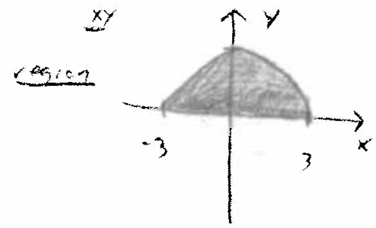
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)dv$$

So, $x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

$x^2+y^2+z^2 = \rho^2$



Since

$z^2 = 9 - x^2 - y^2$ ∴

$x^2+y^2+z^2 = 9$, the region we are integrating over is the top quater portion of a sphere contained in octants 1 and 2.

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^3 \rho^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

Answer $\int_0^{\pi}\int_0^{\pi/2}\int_0^3 \rho^4\sin\phi\,d\rho\,d\phi\,d\theta$

9. Evaluate the inner two integrals for the integral below

$$\int_0^1\int_0^x\int_0^y\int_0^z w\,dw\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x\int_0^y \frac{1}{2}w^2\Big|_0^z\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x\int_0^y \frac{1}{2}z^2\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x \frac{1}{6}z^3\Big|_0^y\,dy\,dx$$

$$= \int_0^1\int_0^x \frac{1}{6}y^3\,dy\,dx$$

Answer $\int_0^1\int_0^x \frac{1}{6}y^3\,dy\,dx$

How to get the Jacobian

$$\left|\frac{\delta(x,y)}{\delta(u,v)}\right| = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = |(1)(-2) - (3)(2)| = |-2 - 6|$$
$$= |-8| = 8$$

where:
$x = u + 2v$
$y = 3u - 2v$

7) continued

So, the following four equations describe the boundary for the region,

① $y = 3x$
② $y = -x + 4$
③ $y = 3x - 8$
④ $y = -x$

and we are told that
$x = u + 2v$
$y = 3u - 2v$

For ① we have

$3u - 2v = 3(u + 2v)$
$\cancel{3u} - 2v = \cancel{3u} + 6v$
$0 = 8v$
$v = 0$

For ② we have

$3u - 2v = -(u + 2v) + 4$
$3u - \cancel{2v} = -u - \cancel{2v} + 4$
$4u = 4$
$u = 1$

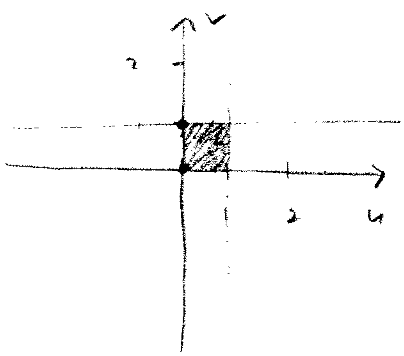
For ③ we have

$3u - 2v = 3(u + 2v) - 8$
$\cancel{3u} - 2v = \cancel{3u} + 6v - 8$
$0 = 8v - 8$
$8 = 8v$
$v = 1$

For ④ we have

$3u - 2v = -(u + 2v)$
$3u - \cancel{2v} = -u - \cancel{2v}$
$4u = 0$
$u = 0$

So, in the uv plane we have.



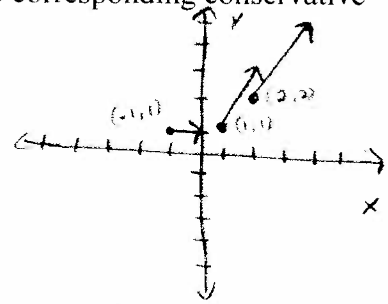
region in UV plane

So, we have $\iint_R (3x - 2y)\,dA =$

$$\int_0^1 \int_0^1 3(u + 2v) - 2(3u - 2v) \left|\frac{\delta(x,y)}{\delta(u,v)}\right| du\,dv$$
$$= \int_0^1 \int_0^1 (3u + 6v - 6u + 4v)(8)\,du\,dv$$
$$= \int_0^1 \int_0^1 (-3u + 10v)(8)\,du\,dv = \int_0^1 \int_0^1 -24u + 80v\,du\,dv$$

[go back to front]

10. Given the potential function f (x,y)= xy + y, graph the vectors for the corresponding conservative vector field from this potential function at points (1,1), (2,2) and (-1,1).



So $f(x,y) = xy + y$

$$\nabla f = \langle f_x, f_y \rangle = \langle y, (x+1) \rangle = \vec{F}$$

$\vec{F}(1,1) = \langle 1, 2 \rangle$

$\vec{F}(2,2) = \langle 2, 3 \rangle$

$\vec{F}(-1,1) = \langle 1, 0 \rangle$

Evaluate the line integral that travels from the origin, across the x-axis, then around the circle of radius 7, up to the point (0,7).

Since it has been stated $\vec{F}$ is a conservative vector field, we have: [it can also easily be shown to be true]

$$\int_C \vec{F} \cdot dr = \int_C \nabla f \cdot dr$$

Answer 7

$$= f(0,7) - f(0,0) = 7 - 0 = 7$$

11. Find the line integral under the function f(x,y) = 2x – 3y and over the semicircle from (-1,0) to (1,0)

$x = \cos t$ where $0 \le t \le \pi$
$y = \sin t$

So, $$\iint_C f(x,y)\, ds = \int_{\pi}^{2\pi} 2\cos t - 3\sin t \sqrt{\sin^2 t + \cos^2 t}\; dt$$

$$\left(\frac{dx}{dt}\right)^2 = (-\sin t)^2 = \sin^2 t$$

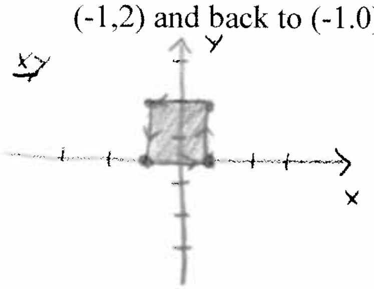
$$\left(\frac{dy}{dt}\right)^2 = (\cos t)^2 = \cos^2 t$$

$$= \int_{\pi}^{2\pi} 2\cos t - 3\sin t\; dt$$

Answer $\int_{\pi}^{2\pi} 2\cos t - 3\sin t\; dt$

12. Find the line integral $\int_C (x+3y)dx + (2x-y)dy$ over the square curve C from (-1,0) to (1,0) to (1,2) to (-1,2) and back to (-1.0) using Green's Theorem

xy



positively oriented

Let $P = x + 3y$

$Q = 2x - y$

So that we have $\int_C P\,dx + Q\,dx$

$$\frac{\delta Q}{\delta x} = 2 \; ; \quad \frac{\delta P}{\delta y} = 3$$

By Greens theorem the line integral is equal to

$$\int_{-1}^{1}\int_{0}^{2} (2-3)\, dy\, dx = \int_{-1}^{1}\int_{0}^{2} -\, dy\, dx$$

Answer $-\int_{-1}^{1}\int_{0}^{2} dy\, dx$

where $-\int_{-1}^{1}\int_{0}^{2} dy\, dx$

$= -4$